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HORIZONTAL ELECTRIC DOUBLET IN A SEMI-INFINITE DISSIPATIVE MEDI--ETC(U)

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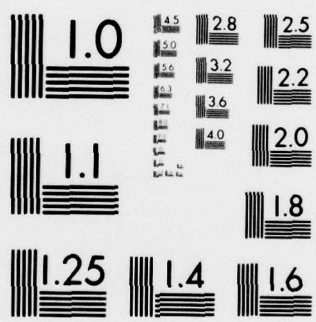

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U.S. Navy Electronics Laboratory

INFORMAL REPORT

Horizontal Electric Doublet in a Semi-Infinite Dissipative Medium

by

R. H. Lien 4/19/51

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U. S. Navy Electronics Laboratory
San Diego, California

INFORMAL REPORT

19 April 1951

HORIZONTAL ELECTRIC DOUBLET IN A SEMI-INFINITE DISSIPATIVE MEDIUM

R. H. LIEN

The units in this report are expressed in the M.K.S. system. The permeability of both media

$$\mu = 4\pi \times 10^{-7} \qquad \text{henry/meter}$$

the dielectric constant of air

$$\epsilon = \frac{1}{36\pi} \times 10^{-9} \qquad \text{farad/meter}$$

The dielectric constant of the dissipative medium

$$\epsilon_d = k\epsilon \qquad \text{farad/meter}$$

where k, a number, is the ratio of the dielectric constant in the medium to that in air.

The conductivity of the dissipative medium is

$$\sigma \qquad \text{mhos/meter}$$

The quantity,

$$\frac{1}{\sqrt{\mu\epsilon}} = c = 3 \times 10^{8} \qquad \text{meters/sec.}$$

In air Maxwell's two curl equations are

(1) $\nabla \times \bar{E} = -j\omega\mu\bar{H}$ $\qquad \bar{E}$ are the electric field vectors

$\nabla \times \bar{H} = j\omega\epsilon\bar{E}$ $\qquad \bar{H}$ are the magnetic field vectors

$\omega = 2\pi f$

f is the frequency in cycles per second.

In the dissipative medium, these equations are

(2) $\nabla \times \bar{E} = -j\omega\mu\bar{H}$

$\nabla \times \bar{H} = j\omega k\epsilon\bar{E} + \sigma\bar{E}$

The second equation of (2) can be modified to read

$$\nabla \times \bar{H} = j\omega\epsilon\tau^2\bar{E};$$

where

$$(3) \quad \tau^2 = K - \frac{j\sigma}{\omega\epsilon}$$

Thus

$$(4) \quad \tau = \beta - j\alpha$$

where

$$\alpha = \sqrt{\frac{K}{2}} \left\{ \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon K}\right)^2} - 1 \right\}^{1/2}$$

$$(5) \quad \beta = \sqrt{\frac{K}{2}} \left\{ \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon K}\right)^2} + 1 \right\}^{1/2}$$

In air the wave number is given by

$$(6) \quad k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

in the dissipative medium the wave number

$$(7) \quad k_d = \omega\sqrt{\mu\epsilon}\,\tau = k\tau$$

In the case of low frequencies $\left(\frac{\sigma}{\omega\epsilon K}\right)^2 >> 1$

$$\alpha \cong \frac{1}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\epsilon}}$$

$$\beta \cong \frac{1}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\epsilon}}$$

$$\alpha \cong \beta$$

In air, the wave length of a plane wave, is written in the form

$e^{-jkz} = e^{-j2\pi\left(\frac{z}{\lambda}\right)}$ where λ is the wave length in air.

In the dissipative medium we have,

$$e^{-jk\tau z} = e^{-j\frac{2\pi}{\lambda}(\alpha - j\alpha)z} = e^{-\frac{2\pi\alpha}{\lambda}z}\, e^{-j\frac{2\pi\alpha z}{\lambda}}$$

In the phase term of the product, let us examine

$$\frac{2\pi}{\lambda_d} = \frac{2\pi\alpha}{\lambda}$$

or

$$\lambda_d = \frac{\lambda}{\alpha} = 2\sqrt{\frac{\pi}{f\mu\sigma}}$$

The phase velocity for a plane wave in air is

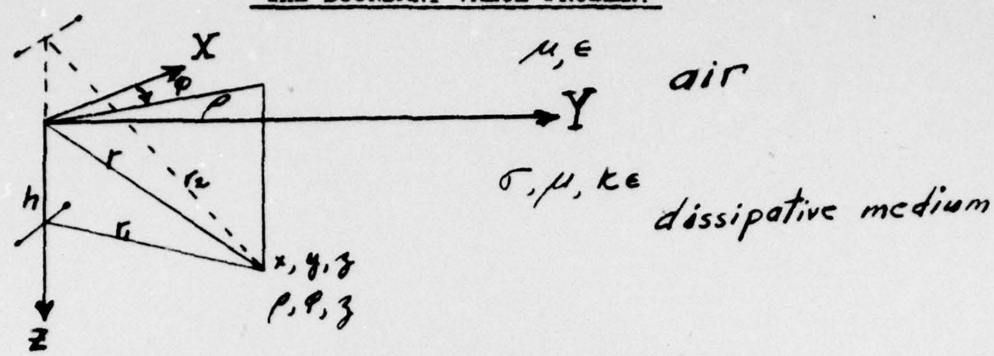
$$\frac{k}{\omega} = \frac{1}{\sqrt{\mu\epsilon}} = c$$

The phase velocity for a plane wave in a dissipative medium is

$\dfrac{1}{c_d} = \dfrac{k\alpha}{\omega}$ whence $c_d = \sqrt{\dfrac{2\omega}{\mu\sigma}} = 2\sqrt{\dfrac{\pi f}{\mu\sigma}}$

HORIZONTAL ELECTRIC DOUBLET IN A SEMI-INFINITE DISSIPATIVE MEDIUM

THE BOUNDARY VALUE PROBLEM



Let a horizontal electric doublet be placed at o, o, h, parallel to the X axis. In this problem it is convenient to use the Hertzian potentials $\bar{\Pi}$ to determine the fields. Two Hertzian potentials $\bar{\Pi}_a$ and $\bar{\Pi}_d$ must be determined (1), (2)

(1) $\nabla^2 \bar{\Pi}_d + k^2 \tau^2 \bar{\Pi}_d = 0$ except at (o, o, h)

$$\bar{E}_d = k^2 \tau^2 \bar{\Pi}_d + \nabla(\nabla \cdot \bar{\Pi}_d)$$

check this in Stratton and/or Sommerfeld.

$$\bar{H}_d = j\omega\epsilon\tau^2 \nabla \times \bar{\Pi}_d$$

and

(2) $\nabla^2 \bar{\Pi}_a + k^2 \bar{\Pi}_a = 0$

$$\bar{E}_a = k^2 \bar{\Pi}_a + \nabla(\nabla \cdot \bar{\Pi}_a)$$

$$\bar{H}_a = j\omega\epsilon \nabla \times \bar{\Pi}_a$$

The boundary condition, that the tangential components of $\bar{E}$ and $\bar{H}$ are continuous across the boundary, imposes the following conditions on $\bar{\Pi}_d$; where

$$\bar{\Pi}_d = \Pi_{xd}\bar{a}_x + \Pi_{zd}\bar{a}_z , \quad \Pi_y = 0$$

when $z = 0$

(3) $\tau^2 \Pi_{xd} = \Pi_{xa}$

(4) $\tau^2 \dfrac{\partial \Pi_{xd}}{\partial z} = \dfrac{\partial \Pi_{xa}}{\partial z}$

(1) Stratton "Electromagnetic Theory", 1941 pp. (573-587)
(2) Sommerfeld "Partial Differential Equations", pp. (236-279)

and

(5) $\tau^2 \Pi_{zd} = \Pi_{za}$

(6) $\dfrac{\partial \Pi_{xd}}{\partial x} + \dfrac{\partial \Pi_{zd}}{\partial z} = \dfrac{\partial \Pi_{xa}}{\partial x} + \dfrac{\partial \Pi_{za}}{\partial z}$

A solution of the scalar wave equation, in cylindrical coordinates, in which $\dfrac{\partial}{\partial \varphi} = 0$ can be written

$$J_0(\xi\rho)e^{-\sqrt{\xi^2-k^2\tau^2}\,z}$$

ξ is a separation constant,

where the sign of the radical is chosen so the quantity vanishes as $z \to \pm\infty$.

Now express

(7) $\Pi_{xd} = \dfrac{I d\hat{x}}{j\omega\epsilon\tau^2 4\pi}\left\{\dfrac{e^{-jk\tau r_1}}{r_1} + \int_0^\infty f_d(\xi)J_0(\xi\rho)e^{-\sqrt{\xi^2-k^2\tau^2}(z+h)}\,\xi d\xi\right\}$

(8) $\Pi_{xa} = \dfrac{I d\hat{x}}{j\omega\epsilon 4\pi}\int_0^\infty f_a(\xi)J_0(\xi\rho)e^{\sqrt{\xi^2-k^2}(z-h)}\,\xi d\xi$

(9) $\Pi_{zd} = \dfrac{I d\hat{x}\cos\varphi}{j\omega\epsilon\tau^2 4\pi}\int_0^\infty g_d(\xi)\,J_1(\xi\rho)e^{-\sqrt{\xi^2-k^2\tau^2}(z+h)}\,\xi d\xi$

(10) $\Pi_{za} = \dfrac{I d\hat{x}\cos\varphi}{j\omega\epsilon 4\pi}\int_0^\infty g_d(\xi)J_1(\xi\rho)e^{\sqrt{\xi^2-k^2}(z-h)}\,\xi d\xi$

Is this basically a guess at the form of the solution? Isn't $e^{-jk\tau r_1}$ missing a something?

Where $(I d\hat{x})$ is the doublet moment; $f(\xi)$ and $g(\xi)$ are functions to be determined from boundary conditions. The source function can be expressed as a Sommerfeld integral.

(11)
$$\frac{e^{-jk\tau r_1}}{r_1} = \begin{cases} \displaystyle\int_0^\infty \frac{J_0(\xi\rho)e^{-\sqrt{\xi^2-k^2\tau^2}(z-h)}\,\xi d\xi}{\sqrt{\xi^2-k^2\tau^2}}, & z > h \\ \displaystyle\int_0^\infty \frac{J_0(\xi\rho)e^{\sqrt{\xi^2-k^2\tau^2}(z+h)}\,\xi d\xi}{\sqrt{\xi^2-k^2\tau^2}}, & z < h \end{cases}$$

Is $r_1 \leftrightarrow \rho$ here? If so, what does this approx. mean?

If we write

(12) $l = \sqrt{\xi^2 - k^2}$ $\qquad m = \sqrt{\xi^2 - k^2\tau^2}$

and apply (11) to (7), and substitute (7) and (8) into (3) and (4), and using the theorem that the Fourier-Bessel transforms are unique,

the following linear simultaneous equations are the results:

$$\frac{e^{-mh}}{m} + f_d\, e^{-mh} = f_a\, e^{-\ell h}$$

$$e^{-mh} - m f_d\, e^{-mh} = \ell f_a\, e^{-\ell h}$$

The solution of these equations results in

$$(13)\quad f_a = \frac{2e^{(\ell-m)h}}{\ell+m}$$

$$(14)\quad f_d = \left(\frac{m-\ell}{\ell+m}\right)\frac{1}{m} = -\frac{1}{m} + \frac{2}{m+\ell}$$

Hence we have

$$(15)\quad \Pi_{xd} = \frac{(Id\hat{x})}{j\omega\epsilon\tau^2 4\pi}\left\{\frac{e^{-jk\tau r_1}}{r_1} - \frac{e^{-jk\tau r_2}}{r_2} + 2\int_0^\infty \frac{J_0(\xi\rho)e^{-m(z+h)}\xi\, d\xi}{\ell+m}\right.$$

$$(16)\quad \Pi_{xo} = \frac{2(Id\hat{x})}{j\omega\epsilon 4\pi}\int_0^\infty \frac{J_0(\xi\rho)e^{-\ell z - mh}\xi\, d\xi}{\ell+m}$$

$$r_1 = \sqrt{\rho^2+(z-h)^2} \qquad\qquad r_2 = \sqrt{\rho^2+(z+h)^2}$$

The operator $\frac{\partial}{\partial x}$ applied to $J_0(\xi\rho)$ results in

$$(17)\quad \frac{\partial}{\partial x} J_0(\xi\rho) = J_0'(\xi\rho)\xi\frac{\partial\rho}{\partial x} = -J_1(\xi\rho)\xi\cos\varphi$$

Using (5), (6), (9), (10), (15), (16), and (17)

results in the equations $g_a = g_d\, e^{-mh+\ell h}$

$$\frac{2}{\tau^2}\frac{e^{-mh}}{\ell+m} + \frac{m g_d}{\tau^2} = \frac{2e^{-mh}}{\ell+m} - g_a \ell e^{-\ell h}$$

The solutions of these equations are

$$(18)\quad g_d = \frac{(1-\tau^2)(-2)}{(\ell+m)(m+\tau^2\ell)}$$

$$(19)\quad g_a = \frac{-2(1-\tau^2)e^{-mh+\ell h}}{(\ell+m)(m+\tau^2\ell)}$$

$$\Pi_{zd} = \frac{(Id\hat{x})}{j\omega\epsilon\tau^2 4\pi}\left\{-2(1-\tau^2)\cos\varphi\int_0^\infty \frac{J_1(\xi\rho)e^{-m(z+h)}\xi^2 d\xi}{(\ell+m)(\tau^2\ell+m)}\right.$$

Now to summarize our position, we have

$$\overline{\Pi_d} = \frac{(I d\hat{x})}{j\omega\epsilon\tau^2 4\pi}\left\{(\psi_1 - \psi_2 + U)\,\bar{a}_x + W\,\bar{a}_z\right\}$$

$$\psi_1 = \frac{e^{-jk\tau r_1}}{r_1}\,, \quad \psi_2 = \frac{e^{-jk\tau r_2}}{r_2}$$

$$r_1^2 = \rho^2 + (z-h)^2\,, \quad r_2^2 = \rho^2 + (z+h)^2$$

$$W = 2(1-\tau^2)\frac{\partial}{\partial x}\int_0^\infty \frac{J_0(\xi\rho)\,e^{-m(z+h)}\,\xi\,d\xi}{(\ell+m)(\tau^2\ell+m)}$$

$$\ell = \sqrt{\xi^2 - k^2} \qquad m = \sqrt{\xi^2 - k^2\tau^2}$$

$$U = 2\int_0^\infty \frac{J_0(\xi\rho)\,e^{-m(z+h)}\,\xi\,d\xi}{\ell+m}$$

How are E_ρ, E_φ, E_z determined by these Π potential?

and in air

$$\overline{\Pi_a} = \frac{2(I d\hat{x})}{j\omega\epsilon 4\pi}\left\{\int_0^\infty \frac{J_0(\xi\rho)\,e^{(\ell z - mh)}\,\xi\,d\xi}{\ell+m}\,\bar{a}_x - \cos\varphi(1-\tau^2)\int_0^\infty \frac{J_1(\xi\rho)\,e^{(\ell z - mh)}\,\xi^2\,d\xi}{(\ell+m)(m+\tau^2\ell)}\,\bar{a}_z\right\}$$

ELECTRIC FIELDS IN THE DISSIPATIVE MEDIUM

It is convenient to express the fields in cylindrical coordinates. This will be accomplished following a method used by K. A. Norton[3].

(1) $$W = 2(1-\tau^2)\frac{\partial}{\partial x}\int_0^\infty \frac{\tau J_0(\xi\rho)\, e^{-m(z+h)}\,\xi\, d\xi}{(l+m)(\tau^2 l+m)}$$

The denominator can be written as

$$\frac{1}{(\tau^2 l+m)(l+m)} = \frac{-\tau^2}{(1-\tau^2)(\tau^2 l+m)\,m} + \frac{1}{(1-\tau^2)(l+m)\,m}$$

Thus equation (1) becomes

(2) $$W = 2\frac{\partial}{\partial x}\int_0^\infty \left\{\frac{1}{m(l+m)} - \frac{\tau^2}{m(\tau^2 l+m)}\right\} J_0(\xi\rho)\, e^{-m(z+h)}\,\xi\, d\xi$$

and

(3) $$\frac{\partial W}{\partial z} = 2\frac{\partial}{\partial x}\int_0^\infty \left\{-\frac{1}{l+m} + \frac{\tau^2}{\tau^2 l+m}\right\} J_0(\xi\rho)\, e^{-m(z+h)}\,\xi\, d\xi$$

If, V is defined as

(4) $$V = 2\tau^2\int_0^\infty \frac{J_0(\xi\rho)\, e^{-m(z+h)}\,\xi\, d\xi}{\tau^2 l+m}$$

then

(5) $$\frac{\partial W}{\partial z} = -\frac{\partial U}{\partial x} + \frac{\partial V}{\partial x}$$

Thus

(6) $$\nabla\cdot\bar{\Pi}_d = \frac{I d\hat{x}}{j\omega\epsilon\tau^2 4\pi}\left\{\frac{\partial}{\partial x}(\psi_1 - \psi_2 + U) - \frac{\partial U}{\partial x} + \frac{\partial V}{\partial x}\right\}$$

or $$\nabla\cdot\bar{\Pi}_d = \frac{I d\hat{x}}{j\omega\epsilon\tau^2 4\pi}\left\{\frac{\partial}{\partial x}(\psi_1 - \psi_2 + V)\right\}$$

Hence the electric fields are, with the factor $\left(\frac{I d\hat{x}}{j\omega\epsilon\tau^2 4\pi}\right)$ understood.

(7) $$E_x = k^2\tau^2(\psi_1 - \psi_2 + U) + \frac{\partial^2}{\partial x^2}(\psi_1 - \psi_2 + V)$$

(8) $$E_y = \frac{\partial^2}{\partial y\,\partial x}(\psi_1 - \psi_2 + V)$$

(9) $$E_z = k^2\tau^2 W + \frac{\partial^2}{\partial z\,\partial x}(\psi_1 - \psi_2 + V)$$

(3) K. A. Norton, "Proc. Inst. Radio Engrs." 24, 1367, 1936; 25, 1203, 1937

From equation (2), and using the identity

$$\frac{\partial}{\partial x} = \cos\varphi \frac{\partial}{\partial \rho}$$ equation (9) becomes

(10) $$E_z = \cos\varphi \left[\frac{\partial^2}{\partial z \partial \rho}(\psi_1 - \psi_2) + 2\frac{\partial}{\partial \rho}\int_0^\infty \left\{\frac{k^2\tau^2}{m(\ell+m)} - \frac{k^2\tau^4}{m(\tau^2\ell+m)} - \frac{m\tau^2}{\tau^2\ell+m}\right\} J_0(\xi\rho) e^{-m(z+h)} \xi d\xi\right.$$

upon writing out the integrals.

The factor

$$\frac{k^2\tau^2}{m(\ell+m)} - \frac{k^2\tau^4}{m(\tau^2\ell+m)} - \frac{m\tau^2}{\tau^2\ell+m} = \frac{-\ell\tau^2}{\tau^2\ell+m} = -\left\{1 - \frac{m}{\tau^2\ell+m}\right\}$$

Thus integral can be written

$$-2\frac{\partial}{\partial\rho}\int_0^\infty \left(1 - \frac{m}{\tau^2\ell+m}\right) J_0(\xi\rho) e^{-m(z+h)} \xi d\xi = 2\frac{\partial^2\psi_2}{\partial z\partial\rho} - \frac{1}{\tau^2}\frac{\partial^2 V}{\partial z\partial\rho}$$

(12) $$E_z = \cos\varphi \frac{\partial^2}{\partial z\partial\rho}\left(\psi_1 + \psi_2 - \frac{V}{\tau^2}\right)$$

Using the identities

$$E_\rho = E_x\cos\varphi + E_y\sin\varphi$$

$$E_\varphi = -E_x\sin\varphi + E_y\cos\varphi$$

and the operators,

$$\frac{\partial}{\partial x} = \cos\varphi\frac{\partial}{\partial\rho}$$

$$\frac{\partial}{\partial y} = \sin\varphi\frac{\partial}{\partial\rho}$$

$$\frac{\partial^2}{\partial x^2} = \cos^2\varphi\frac{\partial^2}{\partial\rho^2} + \frac{\sin^2\varphi}{\rho}\frac{\partial}{\partial\rho}$$

$$\frac{\partial^2}{\partial x\partial y} = \sin\varphi\cos\varphi\frac{\partial^2}{\partial\rho^2} - \frac{\sin\varphi\cos\varphi}{\rho}\frac{\partial}{\partial\rho}$$

it can be shown that

$$E_\rho = \cos\varphi\left\{k^2\tau^2(\psi_1 - \psi_2 + U) + \frac{\partial^2}{\partial\rho^2}(\psi_1 - \psi_2 + V)\right\}$$

$$E_\varphi = \sin\varphi\left\{k^2\tau^2(\psi_1 - \psi_2 + U) + \frac{1}{\rho}\frac{\partial}{\partial\rho}(\psi_1 - \psi_2 + V)\right\}$$

THE EVALUATION OF V FOR LOW FREQUENCIES

The direct evaluation of this integral is difficult. For low frequencies [4], tractable approximations can be obtained. Before performing the integration replace $\xi = k\xi'$; $k\rho = \rho'$; $k\zeta = \zeta'$

Check this

and dropping primes

$$V = 2k\tau^2\int_0^\infty \frac{J_0(\xi\rho)e^{-\zeta\sqrt{\xi^2-\tau^2}}\,\xi\,d\xi}{\tau^2\sqrt{\xi^2-1}+\sqrt{\xi^2-\tau^2}} \qquad \zeta = (z+h)$$

The denominator of the integrand can be simplified as follows:

$$\frac{\tau^2}{\tau^2\sqrt{\xi^2-1}+\sqrt{\xi^2-\tau^2}} = \frac{1}{\sqrt{\xi^2-1}+\sqrt{\frac{\xi^2}{\tau^4}-\frac{1}{\tau^2}}}$$

Let $|\tau^2| \gg 1$, so that $\left|\frac{1}{\tau^2}\right| \cong 0$, whence

$$\frac{\tau^2}{\tau^2\sqrt{\xi^2-1}+\sqrt{\xi^2-\tau^2}} = \frac{1}{\sqrt{\xi^2-1}+\frac{\xi}{\tau^2}} \cong \frac{1}{\sqrt{\xi^2-1}}$$

Hence V reduces to

$$V = 2k\int_0^\infty \frac{J_0(\xi\rho)e^{-\zeta\sqrt{\xi^2-\tau^2}}\,\xi\,d\xi}{\sqrt{\xi^2-1}}$$

$$V = -2k\frac{\partial}{\partial\zeta}\int_0^\infty \frac{J_0(\xi\rho)e^{-\zeta\sqrt{\xi^2-\tau^2}}\,\xi\,d\xi}{\sqrt{\xi^2-1}\,\sqrt{\xi^2-\tau^2}}$$

make a change of variable

$$\eta = \sqrt{\xi^2-\tau^2}$$

and $\sqrt{\xi^2-1} = \sqrt{\eta^2+\tau^2-1} \cong \sqrt{\eta^2+\tau^2}$ since $|\tau^2| \gg 1$;

$$V = -2k\frac{\partial}{\partial\zeta}\int_{j\tau}^\infty \frac{J_0\left(\rho\sqrt{\eta^2+\tau^2}\right)e^{-\eta\zeta}\,d\eta}{\sqrt{\eta^2+\tau^2}}$$

Extending the Laplace - transformation [5], to the complex plane, we have

$$V = -2k\frac{\partial}{\partial\zeta}\left\{I_0\left[\frac{jk\tau}{2}(r_2-z-h)\right]K_0\left[\frac{jk\tau}{2}(r_2+z+h)\right]\right\},$$

where I_0 and K_0 are Modified Bessel Function of zero order.

Using physical units when $\left|\frac{jk\tau}{2}r_2\right| \ll 1$, and using the approximations for small values of the argument in the Bessel Functions.

$$V = \frac{2}{r_2}k$$

---

(4) R. Foster, "Bell System Tech. Journal" 10 July 1931

(5) Magnus and Oberhettinger "Special Functions of Mathematical Physics" p. (133)

When $\left|\frac{jk\tau}{2} r_2\right| >> 1$ and using the asymptotic expansions for the Bessel Functions

$$V = \frac{2k e^{-jk\tau(z+h)}}{\rho}$$

THE EVALUATION OF U FOR LOW FREQUENCIES

The integral representation of U does not lend itself to direct attack, but easily handled approximations can be used for the low frequency case.

$$U = 2k \int_0^\infty \frac{J_0(\xi\rho) e^{-\zeta\sqrt{\xi^2-\tau^2}} \xi d\xi}{\sqrt{\xi^2-1} + \sqrt{\xi^2-\tau^2}}$$

Rationalizing the denominator we have

$$\frac{1}{\sqrt{\xi^2-1} + \sqrt{\xi^2-\tau^2}} = \frac{\sqrt{\xi^2-1} - \sqrt{\xi^2-\tau^2}}{\tau^2-1} \cong \frac{\sqrt{\xi^2-1} - \sqrt{\xi^2-\tau^2}}{\tau^2}$$

which results in

$$U = k\left[\frac{T}{\tau^2} - \frac{2}{\tau^2} \frac{\partial^2 \psi_2}{\partial \zeta^2}\right]$$

where

$$T = 2\int_0^\infty \sqrt{\xi^2-1}\, J_0(\xi\rho) e^{-\zeta\sqrt{\xi^2-\tau^2}} \xi d\xi$$

or

$$T = -\frac{2\partial}{\partial\zeta} \int_0^\infty \frac{\sqrt{\xi^2-1}\, J_0(\xi\rho) e^{-\zeta\sqrt{\xi^2-\tau^2}} \xi d\xi}{\sqrt{\xi^2-\tau^2}}$$

In the second integral make the transformation $\eta = \sqrt{\xi^2-\tau^2}$

and $\sqrt{\xi^2-1} = \sqrt{\eta^2+\tau^2-1} \cong \sqrt{\eta^2+\tau^2}$

$$T = -2\frac{\partial}{\partial\zeta} \int_{j\tau}^\infty \sqrt{\eta^2+\tau^2}\, J_0(\rho\sqrt{\eta^2+\tau^2}) e^{-\eta\zeta} d\eta$$

Using the differential equation for $J_0(x)$, $x = \rho\sqrt{\eta^2+\tau^2}$ we have

$$\frac{1}{\sqrt{\eta^2+\tau^2}} \frac{\partial^2 J_0}{\partial\rho^2} + \frac{1}{\rho\sqrt{\eta^2+\tau^2}} \frac{\partial J_0}{\partial\rho} + \sqrt{\eta^2+\tau^2}\, J_0 = 0$$

This relates T and V as follows

$$T = -\frac{\partial^2 V}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial V}{\partial\rho}$$

$$\therefore U = \frac{1}{\tau^2}\frac{\partial^2 V}{\partial\rho^2} + \frac{1}{\tau^2\rho}\frac{\partial V}{\partial\rho} - \frac{2}{\tau^2}\frac{\partial^2 \psi_2}{\partial\zeta^2}$$

In calculating the electric fields we have to use

$$T + \frac{\partial^2 V}{\partial \rho^2} = -\frac{1}{\rho}\frac{\partial V}{\partial \rho}$$

and

$$T + \frac{1}{\rho}\frac{\partial V}{\partial \rho} = -\frac{\partial^2 V}{\partial \rho^2}$$

DETERMINATION OF THE FIELDS

Substituting the values for U and V and performing the indicated differentiation, and using the physical units involved, we have

$$E_\rho = \frac{(I d\hat{x})\cos\varphi}{4\pi\sigma}\left\{ \begin{aligned} &\psi_1\left[k^2\tau^2\cos^2\theta_1 + \frac{jk\tau(2-3\cos^2\theta_1)}{r_1} + \frac{2-3\cos^2\theta_1}{r_1^2}\right] \\ &+\psi_2\cos^2\theta_2\left[k^2\tau^2 - \frac{3jk\tau}{r_2} - \frac{3}{r_2^2}\right] + T + \frac{\partial^2 V}{\partial\rho^2} \end{aligned} \right\}$$

$$E_\varphi = \frac{-(I d\hat{x})\sin\varphi}{4\pi\sigma}\left\{ \begin{aligned} &\psi_1\left[k^2\tau^2 - \frac{jk\tau}{r_1} - \frac{1}{r_1^2}\right. \\ &+\psi_2\left[-k^2\tau^2 + \frac{jk\tau(3-6\cos^2\theta_2)}{r_2} + \frac{3-6\cos^2\theta_2}{r_2^2}\right] \\ &+ T + \frac{1}{\rho}\frac{\partial V}{\partial\rho} \end{aligned} \right\}$$

$$E_z = \frac{(I d\hat{x})\cos\varphi}{4\pi\sigma}\left\{ \begin{aligned} &\psi_1\left[-k^2\tau^2 + \frac{3jk\tau}{r_1} + \frac{3}{r_1^2}\right]\sin\theta_1\cos\theta_1 \\ &+\psi_2\left[-k^2\tau + \frac{3jk\tau}{r_2} + \frac{3}{r_2^2}\right]\sin\theta_2\cos\theta_2 \\ &- \frac{1}{\tau^2}\frac{\partial^2 V}{\partial z\,\partial\rho} \end{aligned} \right\}$$

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SUMMARY OF FORMULAE FOR ELECTRIC FIELDS

IN A SEMI-INFINITE DISSIPATIVE MEDIUM,

FOR FREQUENCIES LESS THAN 500 KC.

For a horizontal electric dipole placed parallel to the X axis, the X,Y, plane coinciding with the interface and the positive Z axis downward, the expressions for the electric fields in cylindrical coordinates are: (using the MKS system of units)

$$E_\rho = \frac{Il\cos\phi}{4\pi\sigma}\left\{\begin{array}{l} \psi_1\left[k^2\tau^2\cos^2\theta_1 + \dfrac{jk\tau(2-3\cos^2\theta_1)}{r_1} + \dfrac{2-3\cos^2\theta_1}{r_1^2}\right] \\ +\psi_2\cos^2\theta_2\left[k^2\tau^2 - \dfrac{3jk\tau}{r_2} - \dfrac{3}{r_2^2}\right] \\ -\dfrac{1}{\rho}\dfrac{\partial V}{\partial \rho} \end{array}\right\}$$

$$E_\phi = -\frac{Il\sin\phi}{4\pi\sigma}\left\{\begin{array}{l} \psi_1\left[k^2\tau^2 - \dfrac{jk\tau}{r_1} - \dfrac{1}{r_1^2}\right] \\ \psi_2\left[-k^2\tau^2 + \dfrac{jk\tau}{r_2}(3-6\cos^2\theta_2) + \dfrac{1}{r_2^2}(3-6\cos^2\theta_2)\right] \\ -\dfrac{\partial^2 V}{\partial \rho^2} \end{array}\right\}$$

$$E_z = \frac{I l \cos\phi}{4\pi\sigma}\left\{\begin{array}{l} \psi_1\left[-k^2\tau^2 + \dfrac{3jk\tau}{r_1} + \dfrac{3}{r_1^2}\right]\sin\theta_1\cos\theta_1 \\ +\psi_2\left[-k^2\tau^2 + \dfrac{3jk\tau}{r_2} + \dfrac{3}{r_2^2}\right]\sin\theta_2\cos\theta_2 \\ -\dfrac{1}{\tau^2}\dfrac{\partial^2 V}{\partial z\,\partial\rho} \end{array}\right\}$$

Where,

$$\psi_1 = \frac{e^{-jk\tau r_1}}{r_1} \qquad \psi_2 = \frac{e^{-jk\tau r_2}}{r_2}$$

$$r_1^2 = \rho^2 + (z-h)^2 \qquad r_2^2 = \rho^2 + (z+h)^2$$

$$\cos\theta_1 = \frac{z-h}{r_1} \qquad \cos\theta_2 = \frac{z+h}{r_2}$$

z is the depth of the receiver in meters.

h is the depth of the transmitter in meters.

ρ is the horizontal range in meters.

and where,

$$k = \frac{\omega}{c} = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$$

$c = 3 \times 10^8$ meters/sec

$\omega = 2\pi f$ $\quad$ f is cycles/second.

$$\tau^2 = \kappa - \frac{j\sigma}{\omega\epsilon}$$

σ = conductivity in mhos/meter

$\epsilon = \frac{1}{36\pi} \times 10^{-9}$ farads/meter

$\mu = 4\pi \times 10^{-7}$ henries/meter

κ = relative dielectric constant

and

$$\tau = \beta - ja \; ; \quad a = \frac{\sqrt{\kappa}}{\sqrt{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon \kappa}\right)^2} - 1 \right]^{1/2}$$

$$\beta = \frac{\sqrt{\kappa}}{\sqrt{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon \kappa}\right)^2} + 1 \right]^{1/2}$$

For low frequencies $a \cong \beta \cong \frac{1}{\sqrt{2}} \sqrt{\frac{\sigma}{\omega \epsilon}}$

$(I l)$ is the dipole moment

and

$$V = -\frac{\partial}{\partial z} \left\{ I_0 \left[\frac{jk\tau}{2} (r_2 - z - h) \right] K_0 \left[\frac{jk\tau}{2} (r_2 + z + h) \right] \right\} ;$$

Where in I_0 and K_0 are the modified Bessel functions of zero order.

It is of interest now to examine these expressions in the two special cases, viz, $\left| \frac{jk\tau r_2}{2} \right| \ll 1$ and $\left| \frac{jk\tau r_2}{2} \right| \gg 1$.

Case I: $\left| \frac{jk\tau r_2}{2} \right| \ll 1$. Upon using the approximations valid for small values of the variable in the Bessel Function

$$V = \frac{2}{r_2}$$

Thus

$$E_\rho = \frac{Il\cos\phi}{4\pi\sigma}\left\{\begin{aligned} &\psi_1\left[k^2\tau^2\cos^2\theta_1 + \frac{jk\tau(2-3\cos^2\theta_1)}{r_1} + \frac{2-3\cos^2\theta_1}{r_1^2}\right] \\ &+\psi_2\cos^2\theta_2\left[k^2\tau^2 - \frac{3jk\tau}{r_2} - \frac{3}{r_2^2}\right] \\ &+\frac{2}{r_2^3} \end{aligned}\right\}$$

$$E\phi = -\frac{Il\sin\phi}{4\pi\sigma}\left\{\begin{aligned} &\psi_1\left[k^2\tau^2 - \frac{jk\tau}{r_1} - \frac{1}{r_1^2}\right] \\ &+\psi_2\left[-k^2\tau^2 + \frac{jk\tau}{r_2}(3-6\cos^2\theta_2) + \frac{1}{r_2^2}(3-6\cos^2\theta_2)\right] \\ &+\frac{6\cos^2\theta_2-4}{r_2^3} \end{aligned}\right\}$$

$$E_z = \frac{Il\cos\phi}{4\pi\sigma}\left\{\begin{aligned} &\psi_1\left[-k^2\tau^2 + \frac{3jk\tau}{r_1} + \frac{3}{r_1^2}\right]\sin\theta_1\cos\theta_1 \\ &+\psi_2\left[-k^2\tau^2 + \frac{3jk\tau}{r_2} + \frac{3}{r_2^2}\right]\sin\theta_2\cos\theta_2 \\ &-\frac{6}{\tau^2}\,\frac{\sin\theta_2\cos\theta_2}{r_2^3} \end{aligned}\right\}$$

Now when $\omega = 0$, $k\tau = 0$ and $\psi_1 = \frac{1}{r_1}$; $\psi_2 = \frac{1}{r_2}$.

This results in

$$E_\rho = \frac{Il\cos\phi}{4\pi\sigma}\left[\frac{2-3\cos^2\theta_1}{r_1^{\,3}} + \frac{2-3\cos^2\theta_2}{r_2^{\,3}}\right]$$

$$E_\phi = \frac{Il\sin\phi}{4\pi\sigma}\left[\frac{1}{r_1^{\,3}} + \frac{1}{r_2^{\,3}}\right]$$

$$E_z = \frac{Il\cos\phi}{4\pi\sigma}\left[\frac{3\sin\theta_1\cos\theta_1}{r_1^{\,3}} + \frac{3\sin\theta_2\cos\theta_2}{r_2^{\,3}}\right]$$

Now let $\cos\theta_1 \cong \cos\theta_2 \cong 0$ and $r_1 \cong r_2 \cong \rho$

and we have the familiar d.c. case.

$$E_\rho \cong \frac{Il\cos\phi}{\pi\sigma}\;\frac{1}{\rho^3}$$

$$E_\phi \cong \frac{Il\sin\phi}{2\pi\sigma}\;\frac{1}{\rho^3}$$

$$E_z \cong 0$$

Case II: $\left|\frac{jk\tau r_2}{2}\right| \gg 1$, and $\rho \gg (z+h)$.

Upon using the asymptotic expansion of the Bessel Functions, we have

$$V = \frac{2e^{-jk\tau(z+h)}}{\rho}$$

The equations for the fields read

$$E\rho = \frac{Il\cos\phi}{4\pi\sigma}\left\{\begin{aligned} &\psi_1\left[k^2\tau^2 + \frac{jk\tau}{r_1}(2-3\cos^2\theta_1) + \frac{1}{r_1^2}(2-3\cos^2\theta_1)\right] \\ &+\psi_2\cos^2\theta_2\left[k\tau - \frac{3jk\tau}{r_2} - \frac{3}{r_2^2}\right] \\ &+\frac{2e^{-jk\tau(z+h)}}{\rho^3} \end{aligned}\right\}$$

$$E\phi = -\frac{Il\sin\phi}{4\pi\sigma}\left\{\begin{aligned} &\psi_1\left[k^2\tau^2 - \frac{jk\tau}{r_1} - \frac{1}{r_1^2}\right] \\ &+\psi_2\left[-k^2\tau^2 + \frac{jk\tau}{r_2}(3-6\cos^2\theta_2) + \frac{1}{r_2^2}(3-6\cos^2\theta_2)\right] \\ &-\frac{4e^{-jk\tau(z+h)}}{\rho^3} \end{aligned}\right\}$$

$$Ez = \frac{Il\cos\phi}{4\pi\sigma}\left\{\begin{aligned} &\psi_1\left[k^2\tau^2 + \frac{3jk\tau}{r_1} + \frac{3}{r_1^2}\right]\sin\theta_1\cos\theta_1 \\ &+\psi_2\left[-k^2\tau^2 + \frac{3jk\tau}{r_2} + \frac{3}{r_2^2}\right]\sin\theta_2\cos\theta_2 \\ &+\frac{2jk}{\tau}\,\frac{e^{-jk\tau(z+h)}}{\rho^2} \end{aligned}\right\}$$

When $|k\tau r_1| \gg 1$ and when $|k\tau r_2| \gg 1$, then

$\psi_1 \cong \psi_2 \cong 0$ due to the exponential attenuation.

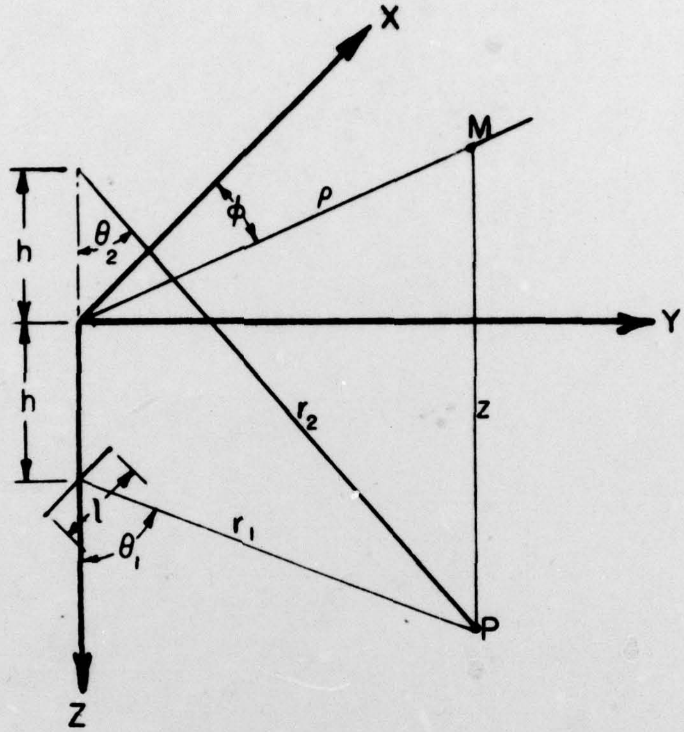
The dominating terms of the fields are

$$E_\rho \cong \frac{Il\cos\phi}{2\pi\sigma} \; \frac{e^{-jk\tau(z+h)}}{\rho^3}$$

$$E_\phi \cong \frac{Il\sin\phi}{\pi\sigma} \; \frac{e^{-jk\tau(z+h)}}{\rho^3}$$

$$E_z \cong \frac{jk}{\tau} \; \frac{Il\cos\phi}{2\pi\sigma} \; \frac{e^{-jk\tau(z+h)}}{\rho^2}$$

Under these conditions there is a change in the E_ρ and E_ϕ as compared with those found for d.c. in Case I.



In ϕ plane